

A FORMULA FOR THE RESOLVENT OF $(-\Delta)^m + M_q^{2m}$ WITH APPLICATIONS TO TRACE CLASS

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ABSTRACT. We derive a formula for the resolvent of the elliptic operator $H = (-\Delta)^m + M_q^{2m}$ on $L_2(\mathbb{R}^N)$ in terms of bounded integral operators S_λ and T_λ whose kernels we know explicitly. We use this formula to specify the domain of the operator $A_\lambda = (H + \lambda I)M_p$ on $L_2(\mathbb{R}^N)$, and to estimate the Hilbert-Schmidt norm of its inverse A_λ^{-1} , for $\lambda \geq 0$. Finally we exploit the last two results to prove a trace class criterion for an integral operator K on $L_2(\mathbb{R}^N)$.

0. Introduction. It is quite common that the resolvent of an elliptic partial differential operator can be represented as an integral operator. Our main result is a simple formula for the resolvent $(H + \lambda I)^{-1}$, $\lambda > 0$, of the selfadjoint elliptic operator $H = (-\Delta)^m + M_q^{2m}$ on $L_2(\mathbb{R}^N)$, where $N \geq 1$ and $m \geq 1$ are integers, Δ is the Laplacian, and M_q denotes the operator of pointwise multiplication by a positive continuous function q on \mathbb{R}^N . We state this formula as a part of our Theorem 1.3. It conveys sufficient information about the kernel of the integral operator $(H + \lambda I)^{-1}$, so that we can decide, after a short computation, whether the inverse A_λ^{-1} of the (nonselfadjoint) closed operator $A_\lambda = (H + \lambda I)M_p$ on $L_2(\mathbb{R}^N)$ is Hilbert-Schmidt or not. Here p is another positive continuous function on \mathbb{R}^N . In this manner we obtain a Hilbert-Schmidt criterion for the operator A_λ^{-1} which we state as Theorem 1.4. To complete our study of the operator A_λ we identify its domain in Theorem 1.5. As an application of Theorems 1.4 and 1.5 we formulate a trace class criterion for an integral operator K on $L_2(\mathbb{R}^N)$ which we state as Theorem 1.6. In this criterion we formulate sufficient conditions on the kernel $k(x, y)$, $x, y \in \mathbb{R}^N$, of the integral operator K which imply that K is of trace class. These conditions require that the kernel $k(x, y)$ have both sufficient smoothness and decay at infinity with respect to the x -variable. As a direct consequence of our Theorem 1.6, we state Corollaries 1.7 and 1.8. The latter one shows overlapping between our results and those of Kamp, Lorentz, and Rejto [9]. Finally we illustrate the optimality of our trace class criterion with Example 1.9.

As for the organization and methods of this paper, we state our main results as Theorems 1.3 through 1.6 in §1.

In §2 we prove Proposition 1.2. To prove the boundedness on $L_2(\mathbb{R}^N)$ of the integral operators S_λ and T_λ from Definition 1.1, we make use of a singular integral method which involves basic facts about the Hardy-Littlewood maximal function.

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The reader is referred to the monograph by Stein [18]. We refer to Takáč [21, Proposition II. 2.3] for an alternative approach where it is proved that the operators S_λ and T_λ are of Holmgren type.

In §3 we prove Theorem 1.3. To prove that the operator $H_0 = (-\Delta)^m + M_q^{2m}$ with domain $\mathcal{D}(H_0) = C_0^\infty(\mathbb{R}^N)$ is essentially selfadjoint on $L_2(\mathbb{R}^N)$, we establish the uniqueness of the solution $u \in L_2(\mathbb{R}^N)$ of the adjoint equation $(H_0^* + \lambda I)u = f$, provided $f \in L_2(\mathbb{R}^N)$ and $\lambda > 0$ sufficiently large are given. This solution has the form $u = (I - T_\lambda)^{-1} S_\lambda f$. We divide the calculation of u into three steps.

In Step 1, in the proof of Proposition 3.2, we approximate the operator $H_0^* + \lambda I$ by a family of operators of the same form with $q(x)^{2m} + \lambda$ replaced by a constant τ^{2m} , where $\tau > 0$. Note that these operators are both translation and rotation invariant. A simple application of the Fourier transformation shows that their inverses can be calculated explicitly in terms of Bessel functions. This approximation idea was suggested by Titchmarsh [23, §17.11, p. 179] in the case $N = 2$ and $m = 1$ when he estimated the Hilbert-Schmidt norm of $(H + \lambda I)^{-1}$.

In Step 2, in the proof of Proposition 3.3, we choose τ to be a function of $x \in \mathbb{R}^N$, $\tau = \tau_\lambda(x) = (q(x)^{2m} + \lambda)^{1/2m}$. Given $f \in L_2(\mathbb{R}^N)$ and $\lambda > 0$ sufficiently large, this choice enables us to compute every solution $u \in L_2(\mathbb{R}^N)$ of the equation $(H_0^* + \lambda I)u = f$ from another equation $u = S_\lambda f + T_\lambda u$.

In Step 3 we finish the proof of the essential selfadjointness of H_0 and obtain the desired formula $(H + \lambda I)^{-1} = (I - T_\lambda)^{-1} S_\lambda$ for the closure H of the operator H_0 , whenever $\lambda > 0$ is sufficiently large.

In §4 we prove Theorem 1.4. To find a necessary and sufficient condition for m, p and q that the operator $A_\lambda^{-1} = M_p^{-1}(H + \lambda I)^{-1} = M_p^{-1} S_\lambda^*(I - T_\lambda^*)^{-1}$ on $L_2(\mathbb{R}^N)$ be Hilbert-Schmidt, we compute the Hilbert-Schmidt norm of the integral operator $M_p^{-1} S_\lambda^*$ (whose kernel we know explicitly) in the proof of Proposition 4.1. Here S_λ^* and T_λ^* denote the adjoints of S_λ and T_λ .

In §5 we prove Theorem 1.5. To specify the domain of the operator H , we prove the boundedness on $L_2(\mathbb{R}^N)$ of the integral operator $M_q^{2m} S_\lambda^*$ in Proposition 5.1. To specify the domain of A_λ , we prove an isomorphism result for two weighted Sobolev spaces which we state as Proposition 5.2.

In §6 we prove Theorem 1.6. We first factorize the integral operator K as $K = A_\lambda^{-1}(A_\lambda K)$ and then exploit the well-known fact that the product of two Hilbert-Schmidt operators, A_λ^{-1} and $A_\lambda K$, is of trace class (cf. Kato [12, Chapter X, §1.3, p. 521]). Namely, the smoothness and decay conditions imposed on the kernel $k(x, y)$ of K guarantee that also the product $A_\lambda K$ is a Hilbert-Schmidt operator on $L_2(\mathbb{R}^N)$.

In §7 we suggest and discuss several possible generalizations of our results (cf. Takáč [21]). We also compare our results to those already known. As for our Theorems 1.3 and 1.4, we refer to Otelbaev [15] (for $N = m = 1$), Titchmarsh [23, §17.11, p. 179] (for $N = 2, m = 1$), and Triebel [24, §6.61, Theorem 1, p. 423] for similar results in the case $p = 1$ on \mathbb{R}^N . These three authors require that $q^{-\gamma} \in L_1(\mathbb{R}^N)$ for some $\gamma > 0$ (in addition to $p = 1$). We do not need this restriction on q , but we impose stronger conditions on the growth of q and its gradient at infinity, i.e. we assume that $\log q$ is uniformly Lipschitz continuous on \mathbb{R}^N . As for our Theorem 1.6 (and Corollaries 1.7 and 1.8), we refer to Lorentz and Rejto [13] and Kamp, Lorentz, and Rejto [9] for similar results in the case $N = 1$,

and p and q have polynomial growth. Another trace class criteria can be found in Stinespring [20], Fabes, Littman, and Riviere [4], Simon [17] and Birman and Solomjak [1, Appendix 4].

In §8 (Appendix) we state two auxiliary results. The first one, Proposition 8.1, is a standard result for the Fourier transform G_{2m} of the analytic function $[(2\pi|y|)^{2m} + 1]^{-1}$ of $y \in \mathbb{R}^N$. The second one, Proposition, 8.2, is a simple technical result.

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1. Main results. We introduce the following notation: $C^k = C^k(\mathbb{R}^N)$ ($k \geq 0$ is an integer or $k = \infty$) is the space of all functions $f: \mathbb{R}^N \rightarrow \mathbb{C}$ which are k -times continuously differentiable; $C_0^k = C_0^k(\mathbb{R}^N)$ is the space of all compactly supported functions from C^k . We endow $C_0^\infty = C_0^\infty(\mathbb{R}^N)$ with the strict inductive limit topology. $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^N)$ is the weak dual of C_0^∞ ; $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ is the space of all rapidly decreasing functions from C^∞ endowed with the Fréchet topology; $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^N)$ is the weak dual of \mathcal{S} . We denote by (\cdot, \cdot) and $\|\cdot\|_{L_2}$ the inner product and norm in $L_2 = L_2(\mathbb{R}^N)$. $B(L_2)$ is the Banach algebra of all bounded linear operators on L_2 endowed with the operator norm $\|\cdot\|_{B(L_2)}$; and $HS(L_2)$ is the space of all Hilbert-Schmidt operators on L_2 endowed with the Hilbert-Schmidt norm $\|\cdot\|_{HS(L_2)}$.

We define the Fourier transformation \mathcal{F} by

$$\mathcal{F}f(x) = \int_{\mathbb{R}^N} e^{-2\pi i xy} f(y) dy, \quad x \in \mathbb{R}^N,$$

for all $f \in L_1(\mathbb{R}^N)$, and extend it to an algebraic and topological isomorphism of \mathcal{S}' onto itself in a unique way (cf. Stein and Weiss [19]).

ASSUMPTIONS ON m, p AND q . Throughout this paper we will assume that $m \geq 1$ is an integer, and p and q are positive functions on \mathbb{R}^N which satisfy $p \in C^{2m}$,

$$(1.1) \quad q_0 = \inf\{q(x) | x \in \mathbb{R}^N\} > 0,$$

and both $\log p$ and $\log q$ are uniformly Lipschitz continuous functions on \mathbb{R}^N with a Lipschitz constant $C > 0$.

We denote by G_{2m} the Fourier transform of the function $[(2\pi|y|)^{2m} + 1]^{-1}$ of $y \in \mathbb{R}^N$. By Proposition 8.1 we have $G_{2m} \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L_1(\mathbb{R}^N)$.

To formulate our results we set

$$(1.2) \quad \tau_\lambda(x) = (q(x)^{2m} + \lambda)^{1/2m}, \quad x \in \mathbb{R}^N, \lambda \geq 0,$$

and introduce the following integral operators:

DEFINITION 1.1. Given $\lambda \geq 0$, we define the functions

$$(1.3) \quad s_\lambda(x, y) = \tau_\lambda(x)^{N-2m} G_{2m}(\tau_\lambda(x)(x - y))$$

and

$$(1.4) \quad t_\lambda(x, y) = s_\lambda(x, y)[q(x)^{2m} - q(y)^{2m}]$$

of the variable $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, $x \neq y$. With these functions we formally associate integral operators S_λ and T_λ on L_2 defined by

$$(1.5) \quad S_\lambda f(x) = \int_{\mathbb{R}^N} s_\lambda(x, y) f(y) dy$$

and

$$(1.6) \quad T_\lambda f(x) = \int_{\mathbb{R}^N} t_\lambda(x, y) f(y) dy,$$

for $f \in C_0^\infty$ and $x \in \mathbb{R}^N$.

These operators have the following properties:

PROPOSITION 1.2. *Let m and q satisfy the assumptions stated above. Then there exists a constant $\lambda_0 > 0$ such that $S_\lambda, T_\lambda \in B(L_2)$ for all $\lambda \geq \lambda_0$, and moreover, there exists a constant $\tilde{C} > 0$ such that*

$$(1.7) \quad \|S_\lambda\|_{B(L_2)} \leq \tilde{C}\lambda^{-1}, \quad \lambda \geq \lambda_0,$$

and

$$(1.8) \quad \|T_\lambda\|_{B(L_2)} \leq \tilde{C}\lambda^{-1/2m}, \quad \lambda \geq \lambda_0.$$

Given a linear operator from a linear space X into another linear space Y , we denote by $\mathbf{D}(A)$ its domain and by $\mathbf{R}(A)$ its range.

Let m and q be as above. We define the operator $H_0 = (-\Delta)^m + M_q^{2m}$ with domain $\mathbf{D}(H_0) = C_0^\infty$. Our main result can be stated as follows:

THEOREM 1.3. *The operator H_0 is a nonnegative and essentially selfadjoint operator on L_2 . If H denotes the (nonnegative selfadjoint) closure of H_0 , then there exists a constant $\lambda_0 > 0$ such that the resolvent of H admits the representation*

$$(1.9) \quad (H + \lambda I)^{-1} = (I - T_\lambda)^{-1} S_\lambda, \quad \lambda \geq \lambda_0,$$

where $(I - T_\lambda)^{-1} \in B(L_2)$.

Let H denote the closure of H_0 on L_2 . Given $\lambda \geq 0$, we define the operator $A_\lambda = (H + \lambda I)M_p$ with domain $\mathbf{D}(A_\lambda) = \{f \in L_2 | pf \in \mathbf{D}(H)\}$.

THEOREM 1.4. *Let m, p and q satisfy the assumptions stated above. Then the operator A_λ on L_2 is densely defined and closed, and has a densely defined and closed inverse A_λ^{-1} , for every $\lambda \geq 0$. This inverse is Hilbert-Schmidt if and only if m, p and q satisfy the conditions*

$$(1.10) \quad 2m > \frac{N}{2} \quad \text{and} \quad \int_{\mathbb{R}^N} p(x)^{-2} q(x)^{N-4m} dx < \infty.$$

Furthermore, if (1.10) is valid, then there exist constants c_1 and c_2 ($0 < c_1 \leq c_2 < \infty$), depending only on $N, m, q_0 = \inf q$ and the Lipschitz constant C for $\log p$ and $\log q$, such that

$$(1.11) \quad c_1 \int_{\mathbb{R}^N} p^{-2} \tau_\lambda^{N-4m} dx \leq \|A_\lambda^{-1}\|_{HS(L_2)}^2 \leq c_2 \int_{\mathbb{R}^N} p^{-2} \tau_\lambda^{N-4m} dx,$$

for all $\lambda \geq 0$.

To specify $\mathbf{D}(A_\lambda)$ we introduce the following spaces: Let w be a positive continuous function on \mathbb{R}^N . We denote by $L_2(w(x) dx)$ the space of all Lebesgue measurable functions $f: \mathbb{R}^N \rightarrow \mathbb{C}$ whose norm

$$\|f\|_{L_2(w(x) dx)} = \left(\int_{\mathbb{R}^N} |f(x)|^2 w(x) dx \right)^{1/2} < \infty,$$

and by $W_2^k(w(x) dx)$ ($k \geq 0$ is an integer) the space of all functions $f \in L_2(w(x) dx)$ whose all distributional derivatives $D^\alpha f$ of order $|\alpha| \leq k$ belong to $L_2(w(x) dx)$, with norm

$$\|f\|_{W_2^k(w(x) dx)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L_2(w(x) dx)}^2 \right)^{1/2}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index with nonnegative integer entries, $|\alpha| = \alpha_1 + \dots + \alpha_N$ and $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}$.

THEOREM 1.5. *Let m, p and q satisfy the assumptions stated above and, in addition, let there exist a constant $C > 0$ such that*

$$(1.12) \quad |D^\alpha p(x)| \leq Cp(x), \quad x \in \mathbb{R}^N,$$

for every multi-index α of order $|\alpha| \leq 2m$. Then $\mathbf{R}(A_\lambda) = L_2$ implies

$$(1.13) \quad \mathbf{D}(A_\lambda) = W_2^{2m}(p(x)^2 dx) \cap L_2(p(x)^2 q(x)^{4m} dx),$$

for all $\lambda \geq 0$. In particular, $\inf\{p(x)^2 q(x)^{4m} | x \in \mathbb{R}^n\} > 0$.

To formulate our trace class criterion we introduce the following spaces of kernels: Again, let w be a positive continuous function on \mathbb{R}^N . We denote by $L_2(w(x) dx dy)$ the space of all Lebesgue measurable functions $f: \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{C}$ whose norm

$$\|f\|_{L_2(w(x) dx dy)} = \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} |f(x, y)|^2 w(x) dx dy \right)^{1/2} < \infty,$$

and by $W_2^{k,0}(w(x) dx dy)$ ($k \geq 0$ is an integer) the space of all functions $f \in L_2(w(x) dx dy)$ whose all distributional derivatives $D_x^\alpha f$ of order $|\alpha| \leq k$ with respect to the x -variable belong to $L_2(w(x) dx dy)$, with norm

$$\|f\|_{W_2^{k,0}(w(x) dx dy)} = \left(\sum_{|\alpha| \leq k} \|D_x^\alpha f\|_{L_2(w(x) dx dy)}^2 \right)^{1/2}.$$

We recall that a compact linear operator K on L_2 is said to be of *trace class* if the sum of all positive eigenvalues (repeated according to their multiplicity) of the nonnegative compact operator $(K^* K)^{1/2}$ converges.

In our next result, K is a Hilbert-Schmidt operator on L_2 of the form

$$Kf(x) = \int_{\mathbb{R}^N} k(x, y) f(y) dy, \quad x \in \mathbb{R}^N, f \in L_2,$$

whose kernel k is in $L_2(\mathbb{R}^N \times \mathbb{R}^N)$.

THEOREM 1.6. *Let m, p and q satisfy the assumptions stated above and, in addition, also (1.10) and (1.12). Let*

$$(1.14) \quad k \in W_2^{2m,0}(p(x)^2 dx dy) \cap L_2(p(x)^2 q(x)^{4m} dx dy).$$

Then $k \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$, and the integral operator K on L_2 with the kernel k is of trace class.

COROLLARY 1.7. *Let $\rho \in C^{2m}$ be a positive function on \mathbb{R}^N which satisfies (1.12) with ρ in place of p . Let us assume that there exists a number $a > 0$ such that*

$$(1.15) \quad \int_{\mathbb{R}^N} \rho(x)^{-a} dx < \infty.$$

Let β and γ be real numbers, and let m, β and γ satisfy

$$(1.16) \quad 2m > \frac{N}{2}, \quad \gamma \geq \frac{a}{2} \quad \text{and} \quad \gamma - \frac{2m}{N}(2\gamma - a) \leq \beta \leq \gamma.$$

Finally, let

$$(1.17) \quad k \in W_2^{2m,0}(\rho(x)^{2\beta} dx dy) \cap L_2(\rho(x)^{2\gamma} dx dy).$$

Then $k \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$, and the integral operator K on L_2 with the kernel k is of trace class.

COROLLARY 1.8. Let β and γ be real numbers, and let m, β and γ satisfy

$$(1.18) \quad 2m > \frac{N}{2}, \quad \gamma > \frac{N}{2} \quad \text{and} \quad \gamma - \frac{2m}{N}(2\gamma - N) < \beta \leq \gamma.$$

Let

$$(1.19) \quad k \in W_2^{2m,0}((1 + |x|^2)^\beta dx dy) \cap L_2((1 + |x|^2)^\gamma dx dy).$$

Then $k \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$, and the integral operator K on L_2 with the kernel k is of trace class.

For $N = 1$, Corollary 1.8 is due to Kamp, Lorentz, and Rejto [9]. More precisely, instead of (1.18) they require that $\alpha = 2m$ be only a real number, and $1/2 < \alpha < 1$, $\beta \geq 0$, $\gamma > 0$ and $\gamma - \alpha(2\gamma - 1) < \beta$. Note that the number α measures fractional smoothness of the kernel $k(x, y)$ with respect to the x -variable. We refer to Takáč [21, Theorem II.2.2] for an analogue of Theorem 1.6 in the case when $\alpha = 2m$ is only a real number.

Finally we illustrate the optimality of our Theorem 1.6 with the following

EXAMPLE 1.9. Let m, p and q satisfy the assumptions stated before Def. 1.1 and, in addition, also (1.12). Let $A_\lambda, \lambda \geq 0$, be the operator on L_2 defined before Theorem 1.4. Then the operator

$$(1.20) \quad K_\lambda = A_\lambda^{-1}(A_\lambda^{-1})^* = M_p^{-1}(H + \lambda I)^{-2}M_p^{-1}$$

exists as a selfadjoint operator on L_2 (cf. Kato [12, Chapter V, Theorem 3.24, p. 275]), because A_λ^{-1} is densely defined and closed on L_2 , by Theorem 1.4. Note that K_λ is possibly unbounded. Let us fix $\lambda \geq 0$.

We claim that the following four statements are equivalent:

- (i) K_λ is of trace class;
- (ii) $A_\lambda^{-1} \in HS(L_2)$;
- (iii) $A_\lambda K_\lambda$ is defined everywhere and $A_\lambda K_\lambda \in HS(L_2)$;
- (iv) m, p and q satisfy (1.10).

Moreover, if any of these four statements is valid, then the kernel k_λ of K_λ satisfies also (1.14).

We postpone also the proofs of Corollaries 1.7, 1.8 and Example 1.9 until after the proof of Theorem 1.6 in §6.

2. Proof of Proposition 1.2. To prove (1.7) and (1.8) we will find suitable majorants for the kernels $s_\lambda(x, y)$ and $t_\lambda(x, y)$, and then apply results from Stein [18] to these majorants. We begin with the following estimate which is an easy

consequence of Proposition 8.1: there exists a constant $c > 0$ such that, for every $\eta \in (0, 2m)$ with $\eta \geq 2m - N$, there exists another constant $C_\eta > 0$ such that

$$(2.1) \quad |G_{2m}(x)| \leq \Phi(x) = C_\eta |x|^{2m-N-\eta} e^{-c|x|}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

We combine (1.3) and (2.1) to obtain

$$(2.2) \quad |s_\lambda(x, y)| \leq C_\eta \tau_\lambda(x)^{-\eta} |x - y|^{2m-N-\eta} e^{-c\tau_\lambda(x)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq 0$. Since $\tau_\lambda(x) \geq \lambda^{1/2m}$ by (1.2), we conclude from (2.2) that

$$(2.3) \quad |s_\lambda(x, y)| \leq \lambda^{-1} \tau_\lambda(x)^N \Phi(\tau_\lambda(x)(x - y)),$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda > 0$. Applying (2.3) to the integral in (1.5) we arrive at

$$(2.4) \quad |S_\lambda f(x)| \leq \lambda^{-1} \sup_{t>0} |\Phi_\tau * f(x)|,$$

for all $x \in \mathbb{R}^N$, $0 \leq f \in L_2$ and $\lambda > 0$, where

$$(2.5) \quad \Phi_\tau(x) = \tau^N \Phi(\tau x), \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \tau > 0.$$

As usual, $*$ denotes convolution. By Stein [18, Chapter III, §2.2, Theorem 2, pp. 62–63] there exists a constant $C' > 0$ such that

$$(2.6) \quad \sup_{\tau>0} |\Phi_\tau * f(x)| \leq C' Mf(x), \quad x \in \mathbb{R}^N, \quad f \in L_2,$$

where Mf denotes the Hardy-Littlewood maximal function associated with f . Again by Stein [18, Chapter I, §1.3, Theorem 1, p. 5] there exists a constant $C'' > 0$ such that

$$(2.7) \quad \|Mf\|_{L_2} \leq C'' \|f\|_{L_2}, \quad f \in L_2.$$

Thus (1.7) follows from a combination of (2.4), (2.6) and (2.7).

To prove (1.8) we first observe that the Lipschitz continuity of $\log q$ on \mathbb{R}^N (with a Lipschitz constant $C > 0$) implies the following two estimates:

$$(2.8) \quad |q(x)^{2m} - q(y)^{2m}| = \left| \int_0^1 \frac{d}{d\xi} q(x + \xi(y-x))^{2m} d\xi \right| \\ \leq 2mC|x-y| \int_0^1 q(x + \xi(y-x))^{2m} d\xi,$$

for all $x, y \in \mathbb{R}^N$, and

$$(2.9) \quad q(y) \leq q(x) e^{C|x-y|}, \quad x, y \in \mathbb{R}^N.$$

We combine (2.8) and (2.9), thus arriving at

$$(2.10) \quad |q(x)^{2m} - q(y)^{2m}| \leq 2mCq(x)^{2m}|x-y|e^{2mC|x-y|},$$

for all $x, y \in \mathbb{R}^N$. Next we conclude from (1.2), (1.4), (2.2) and (2.10) that

$$(2.11) \quad |t_\lambda(x, y)| \leq 2mCC_\eta \tau_\lambda(x)^{2m-\eta} |x-y|^{2m+1-N-\eta} e^{-\theta_\lambda(x)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq 0$, where $\theta_\lambda(x) = c\tau_\lambda(x) - 2mC$. We set $\lambda_0 = (4mC/c)^{2m}$. Then (1.2) implies

$$(2.12) \quad \theta_\lambda(x) \geq \frac{1}{2}c\tau_\lambda(x), \quad x \in \mathbb{R}^N, \quad \lambda \geq \lambda_0.$$

We apply (2.12) to (2.11) to conclude that

$$(2.13) \quad |t_\lambda(x, y)| \leq 2mCC_\eta \tau_\lambda(x)^{2m-\eta} |x-y|^{2m+1-N-\eta} e^{-(c/2)\tau_\lambda(x)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq \lambda_0$. Finally, since $\tau_\lambda(x) \geq \lambda^{1/2m}$ by (1.2), we conclude from (2.13) that

$$(2.14) \quad |t_\lambda(x, y)| \leq \lambda^{-1/2m} \tau_\lambda(x)^N \Psi(\tau_\lambda(x-y)),$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq \lambda_0$, where

$$\Psi(x) = 2mCC_\eta |x|^{2m+1-N-\eta} e^{-(c/2)|x|}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

It is now easy to see that (1.8) follows from (2.14) by the same arguments which we used to derive (1.7) from (2.3).

3. Proof of Theorem 1.3. A $2m$ -fold application of Green's theorem shows that H_0 is a symmetric operator on L_2 . Furthermore, (1.1) yields

$$(3.1) \quad (H_0 u, u) \geq q_0^{2m} \|u\|_{L_2}^2, \quad u \in C_0^\infty.$$

Hence H_0 is nonnegative. In order to prove that H_0 is essentially selfadjoint we need to show that, for some $\lambda > 0$, the operator $H_0^* + \lambda I$ is one-to-one. Here H_0^* denotes the adjoint of H_0 on L_2 . So it suffices to show that there exists a constant $\lambda_0 > 0$ with the following property: given $\lambda \geq \lambda_0$ and $f \in L_2$, the equation

$$(3.2) \quad (-\Delta)^m u + (q^{2m} + \lambda)u = f$$

has a unique solution $u \in L_2$ in the sense of distributions. In the course of the proof of this statement we will derive also (1.9). We divide this proof into the following three steps:

STEP 1. Given $\tau > 0$, let us set

$$(3.3) \quad G_{2m}^{(\tau)}(x) = \tau^{N-2m} G_{2m}(\tau x), \quad x \in \mathbb{R}^N \setminus \{0\}.$$

We note that $G_{2m}^{(\tau)}$ is the Fourier transform of the function $[(2\pi|y|)^{2m} + \tau^{2m}]^{-1}$ of $y \in \mathbb{R}^N$. In this step we will prove the following two results:

LEMMA 3.1. *There exist a constant $\tau_0 > 0$ and a nonnegative function $g \in L_1(\mathbb{R}^N)$ such that*

$$(3.4) \quad |G_{2m}^{(\tau)}(x)| \leq g(x),$$

for all $x \in \mathbb{R}^N \setminus \{0\}$ and $\tau \geq \tau_0$, and

$$(3.5) \quad q(x)^{-2m} |G_{2m}^{(\tau)}(x-y)| q(y)^{2m} \leq g(x-y),$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\tau \geq \tau_0$. In particular, the convolution operator $f \mapsto G_{2m}^{(\tau)} * f$ is bounded on both spaces L_2 and $L_2(q(x)^{-4m} dx)$, for all $\tau \geq \tau_0$.

PROPOSITION 3.2. *Let τ_0 be the constant from Lemma 3.1. Let $\lambda \geq 0$, $u, f \in L_2$ and $\tau \geq \tau_0$ be given. Then equation (3.2) holds in \mathcal{D}' if and only if the equation*

$$(3.6) \quad u = G_{2m}^{(\tau)} * f + G_{2m}^{(\tau)} * [(\tau^{2m} - q^{2m} - \lambda)u]$$

holds in $L_2(q(x)^{-4m} dx)$.

PROOF OF LEMMA 3.1. A combination of (2.1) and (3.3) implies

$$(3.7) \quad |G_{2m}^{(\tau)}(x)| \leq C_\eta \tau^{-\eta} |x|^{2m-N-\eta} e^{-c\tau|x|},$$

$x \in \mathbb{R}^N \setminus \{0\}$, $\tau > 0$. Thus (3.4) follows from (3.7) by fixing an arbitrary $\tau_0 > 0$.

To prove (3.5) we combine (2.9) with (3.7), thus arriving at

$$(3.8) \quad \begin{aligned} q(x)^{-2m} |G_{2m}^{(\tau)}(x-y)| q(y)^{2m} \\ \leq C_\eta \tau^{-\eta} |x-y|^{2m-N-\eta} e^{-(c\tau-2mC)|x-y|}, \end{aligned}$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\tau > 0$. Hence, if we choose $\tau_0 = 4mC/c$, the $\tau \geq \tau_0$ implies $c\tau - 2mC \geq \frac{1}{2}c\tau$, and therefore (3.5) follows from (3.8).

For example, we may choose $\tau_0 = 4mC/c$ and

$$g(x) = C_\eta \tau_0^{-\eta} |x|^{2m-N-\eta} e^{-(c/2)\tau_0|x|}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

to satisfy both (3.4) and (3.5).

The boundedness of the convolution operator $G_{2m}^{(\tau)*}$ on L_2 and $L_2(q(x)^{-4m} dx)$ follows from (3.4) and (3.5), respectively, combined with Young's inequality. The corresponding operator norms of $G_{2m}^{(\tau)*}$ are bounded above by $\int_{\mathbb{R}^N} g(x) dx$, for all $\tau \geq \tau_0$.

PROOF OF PROPOSITION 3.2. Let τ_0 be the constant from Lemma 3.1. Let $\lambda \geq 0$, $u, f \in L_2$ and $\tau \geq \tau_0$ be given. We will prove only that (3.2) implies (3.6), because the converse follows by simply reversing the steps of the first implication.

So let (3.2) be valid. Then we have also

$$[(-\Delta)^m + \tau^{2m} I]u = f + (\tau^{2m} - q^{2m} - \lambda)u,$$

which means

$$(3.9) \quad \langle [(-\Delta)^m + \tau^{2m} I]\phi, u \rangle = \langle \phi, f \rangle + \langle \phi, (\tau^{2m} - q^{2m} - \lambda)u \rangle, \quad \phi \in C_0^\infty,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard duality between C_0^∞ and \mathcal{D}' . Next we observe that the operator $L = (-\Delta)^m + \tau^{2m} I$ is an isomorphism of \mathcal{S} onto itself whose inverse L^{-1} is the convolution operator $L^{-1} = G_{2m}^{(\tau)*}$, (see (3.3) and the remark thereafter). Thus (3.9) implies

$$(3.10) \quad \langle \phi, u \rangle = \langle G_{2m}^{(\tau)*} \phi, f \rangle + \langle G_{2m}^{(\tau)*} \phi, (\tau^{2m} - q^{2m} - \lambda)u \rangle,$$

for all $\phi \in L(C_0^\infty)$. Applying Fubini's theorem to both summands on the right-hand side of (3.10) we obtain

$$\begin{aligned} \langle G_{2m}^{(\tau)*} \phi, f \rangle &= \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} G_{2m}^{(\tau)}(x-y) \phi(y) dy \right] f(x) dx \\ &= \int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^N} G_{2m}^{(\tau)}(y-x) f(x) dx \right] \phi(y) dy = \langle \phi, G_{2m}^{(\tau)} * f \rangle, \end{aligned}$$

and similarly,

$$\langle G_{2m}^{(\tau)*} \phi, (\tau^{2m} - q^{2m} - \lambda)u \rangle = \langle \phi, G_{2m}^{(\tau)} * (\tau^{2m} - q^{2m} - \lambda)u \rangle,$$

for all $\phi \in L(C_0^\infty)$. Note that the function $G_{2m}^{(\tau)}$ is radially symmetric, and that our application of Fubini's theorem is justified by (3.4) and (3.5) combined with $u, f \in L_2$, $\phi \in L(C_0^\infty)$ and Young's inequality. Consequently, (3.10) becomes

$$(3.11) \quad \langle \phi, u \rangle = \langle \phi, G_{2m}^{(\tau)} * f \rangle + \langle \phi, G_{2m}^{(\tau)} * (\tau^{2m} - q^{2m} - \lambda)u \rangle,$$

for all ϕ in $L(C_0^\infty)$. Since $u, f \in L_2$, it follows from Lemma 3.1 that $G_{2m}^{(\tau)} * f \in L_2$ and $G_{2m}^{(\tau)} * (\tau^{2m} - q^{2m} - \lambda)u \in L_2(q(x)^{-4m} dx)$. Note that (1.1) entails that L_2 is continuously imbedded into $L_2(q(x)^{-4m} dx)$. Furthermore, L being an isomorphism of S onto itself, the subspace $L(C_0^\infty)$ is dense in S and consequently also in L_2 . We deduce from (3.11) and the Riesz representation theorem that we must have also $G_{2m}^{(\tau)} * (\tau^{2m} - q^{2m} - \lambda)u \in L_2$, and the equation (3.6) is valid in L_2 . In particular, it is valid also in $L_2(q(x)^{-4m} dx)$. So we have proved Proposition 3.2.

STEP 2. Let τ_0 be the constant from Lemma 3.1. Let $\lambda \geq 0$ and $u, f \in L_2$ be given, and assume that the equation (3.2) is satisfied in \mathcal{D}' . Then, by Proposition 3.2, also (3.6) is valid for all $\tau \geq 0$. In particular, λ, u and f being fixed, there exists a subset $\Omega(\tau) \subset \mathbb{R}^N$ of zero Lebesgue measure such that the equation (3.6) holds pointwise at every point $x \in \mathbb{R}^N \setminus \Omega(\tau)$, whenever $\tau \geq \tau_0$. In (3.6) we want to make a special choice of the parameter τ in terms of $x \in \mathbb{R}^N$, i.e. $\tau = \tau_\lambda(x)$ (see (1.2)). To justify this choice we need to show the following result:

PROPOSITION 3.3. *There exists a constant $\lambda_0 > 0$ with the following property: if $\lambda \geq \lambda_0$ and $u, f \in L_2$ satisfy the equation (3.2) in \mathcal{D}' then the equation*

$$(3.12) \quad u = S_\lambda f + T_\lambda u$$

holds in $L_2(q(x)^{-4m} dx)$.

PROOF. Given $\lambda \geq 0$, let τ_λ be the function defined by (1.2). By our assumptions, q satisfies (1.1) and $\log q$ is uniformly Lipschitz continuous on \mathbb{R}^N . It is easy to verify that also τ_λ satisfies the same assumptions with the same Lipschitz constant $C > 0$ for $\log \tau_\lambda$. Hence we can find a sequence $\{\tau_{\lambda,n} | n = 1, 2, \dots\}$ of Lebesgue measurable functions $\tau_{\lambda,n}$ on \mathbb{R}^N with the following two properties:

- (a) $\tau_{\lambda,n}(\mathbb{R}^N) = \{\tau_{\lambda,n}(x) | x \in \mathbb{R}^N\}$ is a countable subset of $(0, \infty)$, and
- (b) $\tau_\lambda(x)^{2m} \leq \tau_{\lambda,n}(x)^{2m} \leq \tau_\lambda(x)^{2m} + n^{-1}$, $x \in \mathbb{R}^N$, for all $n = 1, 2, \dots$.

For instance, given $n \geq 1$ and $x \in \mathbb{R}^N$, there is a unique integer $k \geq 0$ such that $k/n < \tau_\lambda(x)^{2m} \leq (k+1)/n$. We may set $\tau_{\lambda,n}(x) = [(k+1)/n]^{1/2m}$.

From now on we will assume that $\lambda \geq \lambda_0$, where the constant $\lambda_0 > 0$ is chosen so large that Proposition 1.2 is valid, and also $\lambda_0 \geq \tau_0^{2m}$ where τ_0 is the constant from Lemma 3.1. In particular, (1.2) and (b) imply

$$(3.13) \quad \tau_{\lambda,n}(x) \geq \tau_\lambda(x) \geq \lambda^{1/2m} \geq \tau_0, \quad x \in \mathbb{R}^N.$$

Next let us assume that the equation (3.2) is satisfied in \mathcal{D}' . Consequently, by Proposition 3.2, (3.6) holds in $L_2(q(x)^{-4m} dx)$ for all $\tau \geq \tau_0$. It follows from (a) and (3.13) that we may replace the parameter τ in (3.6) by the function $\tau_{\lambda,n}(x)$, thus obtaining

$$(3.14) \quad u(x) = G_{2m}^{(\tau_{\lambda,n}(x))} * f(x) + G_{2m}^{(\tau_{\lambda,n}(x))} * [(\tau_{\lambda,n}(x)^{2m} - q^{2m} - \lambda)u](x),$$

for all $x \in \mathbb{R}^N \setminus \Omega_{\lambda,n}$, where the set $\Omega_{\lambda,n} = \bigcup \{\Omega(\tau) | \tau \in \tau_{\lambda,n}(\mathbb{R}^N)\}$ has zero Lebesgue measure.

In accordance with Definition 1.1 we introduce the following notation: given $\lambda \geq \lambda_0$ and $n \geq 1$, we define the functions

$$s_{\lambda,n}(x, y) = \tau_{\lambda,n}(x)^{N-2m} G_{2m}(\tau_{\lambda,n}(x)(x - y))$$

and

$$t_{\lambda,n}(x, y) = s_{\lambda,n}(x, y)[\tau_{\lambda,n}(x)^{2m} - q(y)^{2m} - \lambda],$$

of the variable $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, x \neq y$. We denote by $S_{\lambda,n}$ and $T_{\lambda,n}$ the integral operators with the kernel $s_{\lambda,n}(x, y)$ and $t_{\lambda,n}(x, y)$, respectively. Recalling (3.3) we observe that (3.14) reads

$$(3.15) \quad u(x) = S_{\lambda,n}f(x) + T_{\lambda,n}u(x),$$

for a.e. $x \in \mathbb{R}^N$ and $n = 1, 2, \dots$.

Thus equation (3.12) will be verified as soon as we show that the operators $S_{\lambda,n}$ and $T_{\lambda,n}$ are bounded from L_2 into $L_2(q(x)^{-4m} dx)$, and for all $u, f \in L_2$ and $\phi \in L_2(q(x)^{4m} dx)$

$$(3.16) \quad \langle \phi, S_{\lambda,n}f \rangle \rightarrow \langle \phi, S_{\lambda}f \rangle$$

and

$$(3.17) \quad \langle \phi, T_{\lambda,n}u \rangle \rightarrow \langle \phi, T_{\lambda}u \rangle$$

as $n \rightarrow \infty$. Note that, by Proposition 1.2 and (1.1), the operators S_{λ} and T_{λ} are bounded from L_2 into $L_2(q(x)^{-4m} dx)$.

Combining (3.3) and (3.4) with Young's inequality we obtain $S_{\lambda,n} \in B(L_2)$. Thus $S_{\lambda,n}$ is bounded also from L_2 into $L_2(q(x)^{-4m} dx)$, by (1.1). From (b) we obtain

$$q(x)^{-2m}|t_{\lambda,n}(x, y)| \leq (1 + n^{-1}q(x)^{-2m})|s_{\lambda,n}(x, y)| + q(x)^{-2m}|s_{\lambda,n}(x, y)||q(y)^{2m},$$

$x, y \in \mathbb{R}^N, x \neq y$. Combining again (3.3), (3.4) and (3.5) with (1.1) we deduce from the last equality that

$$(3.18) \quad q(x)^{-2m}|t_{\lambda,n}(x, y)| \leq (2 + q_0^{-2m})g(x - y),$$

$x, y \in \mathbb{R}^N, x \neq y$, and $\lambda \geq \lambda_0, n = 1, 2, \dots$. It follows by Young's inequality that $T_{\lambda,n}$ is bounded from L_2 into $L_2(q(x)^{-4m} dx)$.

To prove (3.16) and (3.17) let us fix $u, f \in L_2$, $\phi \in L_2(q(x)^{4m} dx)$ and $\lambda \geq \lambda_0$. First we combine (1.2), (1.3) and (1.4) with (b) to obtain

$$s_{\lambda,n}(x, y) \rightarrow s_{\lambda}(x, y) \quad \text{and} \quad t_{\lambda,n}(x, y) \rightarrow t_{\lambda}(x, y)$$

as $n \rightarrow \infty$, for all $x, y \in \mathbb{R}^N, x \neq y$. Here we have used that $G_{2m} \in C^0(\mathbb{R}^N \setminus \{0\})$. Then we apply Lebesgue's dominated convergence theorem to the sequences of functions

$$\phi(x)s_{\lambda,n}(x, y)f(y) \rightarrow \phi(x)s_{\lambda}(x, y)f(y)$$

and

$$\phi(x)t_{\lambda,n}(x, y)u(y) \rightarrow \phi(x)t_{\lambda}(x, y)u(y)$$

of $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N, x \neq y$, as $n \rightarrow \infty$, thus obtaining (3.16) and (3.17), respectively. In the case of (3.16), a dominating function is provided by $|\phi(x)|g(x - y)|f(y)| \in L_1(\mathbb{R}^N \times \mathbb{R}^N)$ which is a consequence of (3.3), (3.4) and Young's inequality. In the case of (3.17), a dominating function is provided by

$$(2 + q_0^{-2m})|\phi(x)|q(x)^{2m}g(x - y)|u(y)| \in L_1(\mathbb{R}^N \times \mathbb{R}^N)$$

as a consequence of (3.18) and Young's inequality. So we have proved Proposition 3.3.

STEP 3. Let λ_0 be the constant from Proposition 3.3. Let $\lambda \geq \lambda_0$ and $u, f \in L_2$ be given, and assume that equation (3.2) is satisfied in \mathcal{D}' . Then, by Proposition

3.3, also (3.12) is valid. We deduce from (1.8) that we may choose $\lambda_0 > 0$ so large that $\|T_\lambda\|_{B(L_2)} \leq \frac{1}{2}$ whenever $\lambda \geq \lambda_0$. So the inverse $(I - T_\lambda)^{-1}$ exists in $B(L_2)$ for all $\lambda \geq \lambda_0$, and (3.12) implies $u = (I - T_\lambda)^{-1} S_\lambda f$. We conclude that the equation (3.2) has at most one solution $u \in L_2$. Hence H_0 is essentially selfadjoint on L_2 . In particular, the closure H of H_0 coincides with H_0^* , and is selfadjoint. Since H_0 is nonnegative, so is H . Thus $\mathbf{R}(H + \lambda I) = L_2$, for all $\lambda > 0$. It follows that equation (3.2) has at least one solution $u \in L_2$. We conclude that (1.9) is valid.

4. Proof of Theorem 1.4. Since H is the closure in L_2 of the operator $H_0 = (-\Delta)^m + M_q^{2m}$ with the domain $\mathbf{D}(H_0) = C_0^\infty$, we have $C_0^{2m} \subset \mathbf{D}(H)$. Given $\lambda \geq 0$, we combine Theorem 1.3 with (3.1) to conclude that the operator $H + \lambda I$ is one-to-one, and $(H + \lambda I)(C_0^{2m})$ is a dense subspace of L_2 . Furthermore, since p is a positive function from C^{2m} , so is p^{-1} , and M_p is an isomorphism of C_0^{2m} onto itself. Hence the operator A_λ on L_2 is densely defined, one-to-one, and has dense range. In order to show that A_λ is closed it suffices to show that its inverse A_λ^{-1} is closed. Indeed, $A_\lambda^{-1} = M_p^{-1}(H + \lambda I)^{-1}$ is closed because M_p^{-1} is closed and $(H + \lambda I)^{-1}$ is bounded on L_2 .

We divide the proof of the Hilbert-Schmidt criterion for A_λ^{-1} into two steps. In Step 1 we will prove it for all λ sufficiently large. In Step 2 we will extend this result to the case when $\lambda \geq 0$ is arbitrary.

STEP 1. Throughout this step we assume that $\lambda > 0$ is so large that Proposition 1.2 and Theorem 1.3 are valid. We deduce from (1.8) that we may choose λ_0 so large that $\|T_\lambda\|_{B(L_2)} \leq \frac{1}{2}$ whenever $\lambda \geq \lambda_0$. Since H is selfadjoint, (1.9) yields $(H + \lambda I)^{-1} = S_\lambda^*(I - T_\lambda^*)^{-1}$ for every $\lambda \geq \lambda_0$. Hence

$$(4.1) \quad A_\lambda^{-1} = M_p^{-1} S_\lambda^*(I - T_\lambda^*)^{-1}, \quad \lambda \geq \lambda_0.$$

Given $\lambda \geq \lambda_0$, it follows that $A_\lambda^{-1} \in HS(L_2)$ if and only if $M_p^{-1} S_\lambda^* \in HS(L_2)$, and

$$\frac{2}{3} \|M_p^{-1} S_\lambda^*\|_{HS(L_2)} \leq \|A_\lambda^{-1}\|_{HS(L_2)} \leq 2 \|M_p^{-1} S_\lambda^*\|_{HS(L_2)}.$$

We recall from equation (1.5) that $M_p^{-1} S_\lambda^*$ is an integral operator with the kernel $p(x)^{-1} s_\lambda(y, x)$. Hence, in order to prove our Hilbert-Schmidt criterion and (1.11), for all $\lambda \geq \lambda_0$, we need to show only the following result:

PROPOSITION 4.1. *Let m, p and q be as in Theorem 1.4. Let $s_\lambda(x, y)$ be defined by (1.3). Then there exist constants $\lambda'_0 > 0$ and c'_1, c'_2 ($0 < c'_1 \leq c'_2 < \infty$), depending only on N, m and the Lipschitz constant C for $\log p$, with the following properties:*

(a) *Given $\lambda \geq \lambda'_0$, then we have $p(x)^{-1} s_\lambda(y, x) \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$ if and only if m, p and q satisfy (1.10).*

(b) *If (1.10) is valid, then*

$$(4.2) \quad \begin{aligned} c'_1 \int_{\mathbb{R}^N} p^{-2} \tau_\lambda^{N-4m} dx &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} p(x)^{-2} |s_\lambda(y, x)|^2 dx dy \\ &\leq c'_2 \int_{\mathbb{R}^N} p^{-2} \tau_\lambda^{N-4m} dx, \end{aligned}$$

for all $\lambda \geq \lambda'_0$.

PROOF. Assume that $\lambda \geq 1$ and $p(x)^{-1} s_\lambda(y, x) \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$. By (1.3) we have

$$p(x)^{-1} s_\lambda(y, x) = p(x)^{-1} \tau_\lambda(y)^{N-2m} G_{2m}(\tau_\lambda(y)(y - x)),$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$. Using (2.9) with p in place of q we obtain that there is a constant $C' > 0$ (e.g. $C' = e^C$) such that $p(x) \leq C'p(y)$, for all $x, y \in \mathbb{R}^N$ with $|x - y| \leq 1$. Applying this estimate to (4.3) we arrive at

$$(4.4) \quad p(x)^{-1}|s_\lambda(y, x)| \geq (C')^{-1}p(y)^{-1}\tau_\lambda(y)^{N-2m}|G_{2m}(\tau_\lambda(y)(y-x))|,$$

for all $x, y \in \mathbb{R}^N$ with $0 < |x - y| \leq 1$. We set $\Omega = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N | 0 < |x - y| \leq \tau_\lambda(y)^{-1}\}$. Note that $\tau_\lambda(y) \geq \lambda^{1/2m}$, $y \in \mathbb{R}^N$, by (1.2), and $\lambda \geq 1$ by our assumption. Thus $\tau_\lambda(y) \geq 1$, $y \in \mathbb{R}^N$, and consequently (4.4) holds for all $(x, y) \in \Omega$. We first square both sides of (4.4), then integrate over the set Ω , and finally apply Fubini's theorem followed by a substitution $z = \tau_\lambda(y)(y - x)$ for x , with $dz = \tau_\lambda(y)^N dx$, thus obtaining

$$\begin{aligned} (4.5) \quad & \iint_{\Omega} p(x)^{-2}|s_\lambda(y, x)|^2 dx dy \\ & \geq (C')^{-2} \iint_{\Omega} p(y)^{-2}\tau_\lambda(y)^{2N-4m}|G_{2m}(\tau_\lambda(y)(y-x))|^2 dx dy \\ & = (C')^{-2} \int_{|z| \leq 1} |G_{2m}(z)|^2 dz \cdot \int_{\mathbb{R}^N} p(y)^{-2}\tau_\lambda(y)^{N-4m} dy. \end{aligned}$$

Since the last two integrals are nonzero they must converge. Hence we have

$$(4.6) \quad c'_1 = (C')^{-2} \int_{|z| \leq 1} |G_{2m}(z)|^2 dz \in (0, \infty)$$

and

$$(4.7) \quad \int_{\mathbb{R}^N} p(y)^{-2}\tau_\lambda(y)^{N-4m} dy \in (0, \infty).$$

We deduce from (2.1) and (4.6) that $G_{2m} \in L_2(\mathbb{R}^N)$. Hence, by Parseval's identity, the inverse Fourier transform $[(2\pi|y|)^{2m} + 1]^{-1}$ of G_{2m} must be in $L_2(\mathbb{R}^N)$. It follows that $2m > N/2$. Since (1.1) and (1.2) imply $\tau_\lambda(y) \leq (1 + \lambda q_0^{-2m})^{1/2m} q(y)$, $y \in \mathbb{R}^N$, we deduce from (4.7) and $4m > N$ that (1.10) is valid. Furthermore, (4.5) implies the first estimate in (4.2), for all $\lambda \geq 1$.

Assume now that (1.10) is valid. Recalling (2.2) we observe that there exists a constant $c > 0$ such that, for every $\eta \in (0, 2m)$, $\eta \geq 2m - N$, there exists another constant $C_\eta > 0$ such that

$$(4.8) \quad p(x)^{-1}|s_\lambda(y, x)| \leq C_\eta p(x)^{-1}\tau_\lambda(y)^{-\eta}|x - y|^{2m-N-\eta}e^{-c\tau_\lambda(y)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq 0$. We apply (2.9) with p in place of q to (4.8), thus obtaining

$$(4.9) \quad p(x)^{-1}|s_\lambda(y, x)| \leq C_\eta p(y)^{-1}\tau_\lambda(y)^{-\eta}|x - y|^{2m-N-\eta}e^{-\theta_\lambda(y)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq 0$, where $\theta_\lambda(y) = c\tau_\lambda(y) - C$. We set $\lambda'_0 = \max\{1, (2C/c)^{2m}\}$. Then (1.2) implies (2.12) again, for all $\lambda \geq \lambda'_0$. We apply (2.12) to (3.9) to conclude that

$$(4.10) \quad p(x)^{-1}|s_\lambda(y, x)| \leq C_\eta p(y)^{-1}\tau_\lambda(y)^{-\eta}|x - y|^{2m-N-\eta}e^{-(c/2)\tau_\lambda(y)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq \lambda'_0$. We first square both sides of (4.10), then integrate with respect to x over \mathbb{R}^N , and finally substitute $z = c\tau_\lambda(y)(x - y)$ for x , with $dz = (c\tau_\lambda(y))^N dx$, thus arriving at

$$(4.11) \quad \int_{\mathbb{R}^N} p(x)^{-2} |s_\lambda(y, x)|^2 dx \\ \leq C_\eta^2 c^{N+2\eta-4m} \omega_{N-1} \Gamma(4m - N - 2\eta) p(y)^{-2} \tau_\lambda(y)^{N-4m},$$

for all $y \in \mathbb{R}^N$ and $\lambda \geq \lambda'_0$, where ω_{N-1} is the surface area of a unit sphere in \mathbb{R}^N , and the Euler gamma function arises from the integral

$$\int_{\mathbb{R}^N} |z|^{4m-2N-2\eta} e^{-|z|} dz = \omega_{N-1} \Gamma(4m - N - 2\eta).$$

Note that $2m > N/2$ is the first hypothesis in (1.10). Since η has to satisfy only $\eta \in (0, 2m)$ and $\eta \geq 2m - N$, we may choose it such that $\max\{0, 2m - N\} < \eta < 2m - N/2$. Thus the last integral converges. Since (1.2) implies $\tau_\lambda(y) \geq q(y)$, $y \in \mathbb{R}^N$, we deduce from (4.11) and $4m > N$ that $p(x)^{-1} s_\lambda(y, x) \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$. Furthermore, an integration of (4.11) with respect to $y \in \mathbb{R}^N$ implies the second estimate in (4.2).

STEP 2. Let λ_0 be the constant specified at the beginning of Step 1. It follows from the proofs of Proposition 1.2 and Theorem 1.3 that the constants λ_0 and \tilde{C} in these two results depend only on N, m and the Lipschitz constant C for $\log q$. Let λ'_0 be the constant from Proposition 4.1. Hence we may choose λ_0 so large that also $\lambda_0 \geq \lambda'_0$, where λ_0 depends only on N, m and the Lipschitz constant C for $\log p$ and $\log q$. By Step 1, our Hilbert-Schmidt criterion and (1.11) are valid for every $\lambda \geq \lambda_0$. To extend this result to the case when $\lambda \geq 0$ let us consider the resolvent identity for the operator H on L_2 , i.e.

$$(H + \lambda I)^{-1} - (H + \mu I)^{-1} = (\mu - \lambda)(H + \lambda I)^{-1}(H + \mu I)^{-1},$$

$\lambda, \mu \geq 0$. These inverses belong to $B(L_2)$ as a consequence of Theorem 1.3 and (3.1). It follows that

$$A_\lambda^{-1} - A_\mu^{-1} = (\mu - \lambda)A_\lambda^{-1}(H + \mu I)^{-1}, \quad \lambda, \mu \geq 0.$$

Since $(H + \mu I)^{-1} \in B(L_2)$ for $\mu \geq 0$, we deduce that $A_\lambda^{-1} \in HS(L_2)$ for some $\lambda \geq 0$ implies $A_\mu^{-1} \in HS(L_2)$ for all $\mu \geq 0$. This argument extends our Hilbert-Schmidt criterion to all $\lambda \geq 0$, i.e. $A_\lambda^{-1} \in HS(L_2)$ if and only if (1.10) holds. Assume now that (1.10) holds. Thus (1.11) holds for all $\lambda \geq \lambda_0$, by Step 1. The last identity shows that

$$\|A_\mu^{-1}\|_{HS(L_2)} \leq [1 + \lambda_0 \|(H + \lambda I)^{-1}\|_{B(L_2)}] \|A_\lambda^{-1}\|_{HS(L_2)},$$

for all $\lambda, \mu \in [0, \lambda_0]$. Moreover, (3.1) entails $\|(H + \lambda I)^{-1}\|_{B(L_2)} \leq q_0^{-2m}$, $\lambda \geq 0$. Hence

$$\|A_\mu^{-1}\|_{HS(L_2)} \leq (1 + \lambda_0 q_0^{-2m}) \|A_\lambda^{-1}\|_{HS(L_2)},$$

for all $\lambda, \mu \in [0, \lambda_0]$. This inequality extends (1.11) to the case when $\lambda \geq 0$ is arbitrary.

5. Proof of Theorem 1.5. We begin with the proof of the following special case of (1.13):

$$(5.1) \quad \mathbf{D}(H) = W_2^{2m} \cap L_2(q(x)^{4m} dx).$$

Clearly C_0^∞ is dense in both W_2^{2m} and $L_2(q(x)^{4m} dx)$, and consequently also in $\mathcal{W}_{2m} = W_2^{2m} \cap L_2(q(x)^{4m} dx)$ with the natural Hilbert space structure induced by those of W_2^{2m} and $L_2(q(x)^{4m} dx)$. First we observe that H_0 is a bounded linear operator from \mathcal{W}_{2m} into L_2 with dense domain. Hence $\mathcal{W}_{2m} \subset \mathbf{D}(H)$, since $\mathcal{W}_{2m} \subset L_2$ both algebraically and topologically, and H is the closure of H_0 in L_2 . In order to prove that $\mathbf{D}(H) \subset \mathcal{W}_{2m}$ it suffices to prove that there exists a constant $\lambda_0 > 0$ with the following property: given $\lambda \geq \lambda_0$ and $f \in L_2$, every solution $u \in L_2$ of equation (3.2) (in the sense of distributions) satisfies $u \in \mathcal{W}_{2m}$. According to Step 3 in the proof Theorem 1.3 we may choose λ_0 so large that, given $\lambda \geq \lambda_0$ and $f \in L_2$, equation (3.2) has a unique solution $u \in L_2$ in \mathcal{D}' . Moreover, we have $u = (I - T_\lambda)^{-1} S_\lambda f = S_\lambda^* (I - T_\lambda^*)^{-1} f$, $\lambda \geq \lambda_0$. Since by (3.2) we have

$$(5.2) \quad [(-\Delta)^m + \lambda I]u = f - q^{2m}u$$

in \mathcal{D}' , it suffices to show that $q^{2m}u \in L_2$. In fact, then (5.2) entails $u \in W_2^{2m}$ by a simple Fourier transformation argument, and so $u \in \mathcal{W}_{2m}$. To prove that $q^{2m}u = M_q^{2m} S_\lambda^* (I - T_\lambda^*)^{-1} f \in L_2$, we need to prove only the following result:

PROPOSITION 5.1. *Let m and q be as in Theorem 1.5. Then there exists a constant $\lambda_0 > 0$ such that $M_q^{2m} S_\lambda^* \in B(L_2)$ for all $\lambda \geq \lambda_0$, and moreover, there exists a constant $\tilde{C} > 0$ such that*

$$(5.3) \quad \|M_q^{2m} S_\lambda^*\|_{B(L_2)} \leq \tilde{C}, \quad \lambda \geq \lambda_0.$$

PROOF. It follows from (1.5) that $M_q^{2m} S_\lambda^*$ is an integral operator on L_2 with the kernel $q(x)^{2m} s_\lambda(y, x)$. Next we conclude from (1.2) _{λ} (2.2) and (2.9) that

$$(5.4) \quad |q(x)^{2m} s_\lambda(y, x)| \leq C_\eta \tau_\lambda(y)^{2m-\eta} |x-y|^{2m-N-\eta} e^{-\theta_\lambda(y)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq 0$, where $\theta_\lambda(y) = c\tau_\lambda(y) - 2mC$. We set $\lambda_0 = (4mC/c)^{2m}$. Then (1.2) implies (2.12) again, for all $\lambda \geq \lambda_0$. We apply (2.12) to (5.4) to conclude that

$$(5.5) \quad |q(x)^{2m} s_\lambda(y, x)| \leq C_\eta \tau_\lambda(y)^{2m-\eta} |x-y|^{2m-N-\eta} e^{(c/2)\tau_\lambda(y)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq \lambda_0$. We conclude that

$$(5.6) \quad |q(x)^{2m} s_\lambda(y, x)| \leq 2^{2m-N-\eta} \tau_\lambda(y)^N \Phi\left(\frac{1}{2}\tau_\lambda(y)(x-y)\right),$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$, and $\lambda \geq \lambda_0$, where the function Φ has been defined by (2.1). It is now easy to see that (5.3) follows from (5.6) by the same arguments which we have used in the proof of Proposition 1.2 to derive (1.7) from (2.3).

Thus we have verified (5.1). hence, in order to finish the proof of (1.13) we need to prove only the following result:

PROPOSITION 5.2. *Let $k \geq 0$ be an integer, and let p be a positive function on \mathbb{R}^N which satisfies both $p \in C^k$ and (1.12) for every multi-index α of order*

$|\alpha| \leq k$. Let w be a positive continuous function on \mathbb{R}^N . Then M_p is an algebraic and topological isomorphism of $W_2^k(p(x)^2 w(x) dx)$ onto $W_2^k(w(x) dx)$.

PROOF. An elementary computation shows that (1.12) is valid also for p^{-1} in place of p . Clearly $M_p^{-1} = M_{p^{-1}}$ if we conceive M_p and $M_{p^{-1}}$ as linear operators on $L_{2,\text{loc}}(\mathbb{R}^N)$. Hence it suffices to prove that M_p is a bounded linear operator from $W_2^k(p(x)^2 w(x) dx)$ into $W_2^k(w(x) dx)$. In fact, in this result we may replace p by p^{-1} , and w by $p^2 w$ to conclude that M_p^{-1} is a bounded linear operator from $W_2^k(w(x) dx)$ into $W_2^k(p(x)^2 w(x) dx)$.

Since C_0^k is dense in $W_2^k(w(x) dx)$, and M_p is a one-to-one mapping of C_0^k onto itself, the desired boundedness of M_p from $W_2^k(p(x)^2 w(x) dx)$ into $W_2^k(w(x) dx)$ follows from the following claim: there exists a constant $C_k > 0$ such that

$$(5.7) \quad \|p\phi\|_{W_2^k(w(x) dx)} \leq C_k \|\phi\|_{W_2^k(p(x)^2 w(x) dx)},$$

for all $\phi \in C_0^k$.

To prove this claim we take a multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ of order $|\alpha| \leq k$, and $\phi \in C_0^k$. We apply the Leibnitz formula to calculate the partial derivative

$$(5.8) \quad D^\alpha(p\phi) = \sum_{\delta+\varepsilon=\alpha} C(\delta, \varepsilon) D^\delta p \cdot D^\varepsilon \phi.$$

Here $C(\delta, \varepsilon)$ are positive integers which can be calculated explicitly in terms of the multi-indices δ and ε . Note that $C(\delta, \varepsilon) = 1$ for $\delta = (0, \dots, 0)$. We apply (1.12) for δ to (5.8) to obtain a constant $C'_k > 0$ such that

$$|D^\alpha(p\phi)| \leq C'_k p \sum_{\varepsilon \leq \alpha} |D^\varepsilon \phi| \quad \text{on } \mathbb{R}^N,$$

for all $\phi \in C_0^k$. Here $\varepsilon \leq \alpha$ means $\varepsilon_i \leq \alpha_i$ for all $i = 1, \dots, N$. Hence, making use of Cauchy's inequality, we obtain another constant $C''_k > 0$ such that

$$|D^\alpha(p\phi)|^2 \leq C''_k p^2 \sum_{\varepsilon \leq \alpha} |D^\varepsilon \phi|^2 \quad \text{on } \mathbb{R}^N,$$

for all $\phi \in C_0^k$. Finally we integrate this estimate over \mathbb{R}^N with respect to the measure $w(x) dx$, and then sum these integrals up with respect to α , $|\alpha| \leq k$, thus arriving at (5.7).

6. Proof of Theorem 1.6. By Theorem 1.4, (1.10) implies $A_\lambda^{-1} \in HS(L_2)$, for all $\lambda \geq 0$. Note that $\mathbf{R}(A_\lambda) = \mathbf{D}(A_\lambda^{-1}) = L_2$. Hence (1.13) is valid by Theorem 1.5. We set $\mathcal{V}_{2m} = W_2^{2m}(p(x)^2 dx) \cap L_2(p(x)^2 q(x)^{4m} dx)$ with the natural Hilbert space structure induced by those of $W_2^{2m}(p(x)^2 dx)$ and $L_2(p(x)^2 q(x)^{4m} dx)$. By the closed graph theorem, A_λ is a bounded linear operator from \mathcal{V}_{2m} onto L_2 . Thus (1.14) shows that the functions $k(x, y)$ and $b(x, y) = [A_\lambda k(\cdot, y)](x)$ of $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ are in $L_2(\mathbb{R}^N \times \mathbb{R}^N)$, and $b(x, y)$ is the kernel of an integral operator $B \in HS(L_2)$. We claim that, given $\lambda \geq 0$, we have

$$(6.1) \quad \mathbf{R}(K) \subset \mathbf{D}(A_\lambda) \quad \text{and} \quad A_\lambda K = B.$$

In fact, given $f \in L_2$ and $\phi \in \mathbf{D}(A_\lambda^*)$, we employ Fubini's and Green's theorems to obtain

$$\begin{aligned}
 (6.2) \quad \int_{\mathbb{R}^N} \phi(x) Bf(x) dx &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \phi(x) [A_\lambda k(\cdot, y)](x) f(y) dx dy \\
 &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} A_\lambda^* \phi(x) k(x, y) f(y) dx dy \\
 &= \int_{\mathbb{R}^N} A_\lambda^* \phi(x) Kf(x) dx.
 \end{aligned}$$

Since A_λ is closed, its adjoint A_λ^* is densely defined (cf. Kato [12, Chapter III, Theorem 5.29, p. 169]). Thus (6.2) and $Bf \in L_2$ imply $Kf \in \mathbf{D}(A_\lambda^{**})$ and $Bf = A_\lambda^{**} Kf$. Again, since A_λ is closed, we have $A_\lambda^{**} = A_\lambda$. We conclude that (6.1) is valid. In particular, we have $A_\lambda K \in HS(L_2)$ for every $\lambda \geq 0$.

In order to finish our proof we factorize K as $K = A_\lambda^{-1}(A_\lambda K)$, $\lambda \geq 0$. Since both factors are in $HS(L_2)$, we conclude that K is of trace class (cf. Kato [12, Chapter X, §1.3, p. 521]).

PROOF OF COROLLARY 1.7. We set $p = \rho^\beta$ and $q = \rho^{(\gamma-\beta)/2m}$. An easy computation shows that also p and q satisfy (1.12). In particular, both $\log p$ and $\log q$ are uniformly Lipschitz continuous on \mathbb{R}^N . Furthermore, we have $\rho_0 = \inf \rho > 0$, by Proposition 8.2. Thus q satisfies (1.1), since $\gamma \geq \beta$ by (1.16). Finally (1.15) and (1.16) imply (1.10), and (1.17) implies (1.14). The desired conclusion follows now from Theorem 1.6.

PROOF OF COROLLARY 1.8. Let $\rho(x) = (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^N$. Clearly ρ satisfies the hypotheses of Corollary 1.7 with any $a > N$. We choose a such that $\gamma - (2m/N)(2\gamma - a) = \beta$. Then (1.18) implies both (1.15) with $a > N$, and (1.16). Finally (1.19) implies (1.17). Thus we can apply Corollary 1.7 to obtain the desired conclusion.

PROOF OF EXAMPLE 1.9. 1. The equivalence of (i) and (ii) is a direct consequence of the definition of trace class.

2. The equivalence of (ii) and (iii) follows from $\mathbf{R}(K_\lambda) \subset \mathbf{D}(A_\lambda)$ and

$$\text{Graph}(A_\lambda K_\lambda) \subset \text{Graph}((A_\lambda^{-1})^*).$$

3. The equivalence of (ii) and (iv) follows from Theorem 1.4.

4. Let us now assume that these four statements are valid. Let k_λ and b_λ denote the kernels of the integral operators K_λ and $A_\lambda K_\lambda$, respectively. By (i) and (iii) we have $k_\lambda, b_\lambda \in L_2(\mathbb{R}^N \times \mathbb{R}^N)$. The same arguments which we used in the proof of Theorem 1.6, and a calculation similar to (6.2) yield $k_\lambda(\cdot, y) \in \mathbf{D}(A_\lambda)$ and $A_\lambda k_\lambda(\cdot, y) = b_\lambda(\cdot, y)$, for a.e. $y \in \mathbb{R}^N$. By (ii), we can apply Theorem 1.5 to conclude that the kernel k_λ satisfies (1.14), since $k_\lambda(\cdot, y) = A_\lambda^{-1} b_\lambda(\cdot, y)$, for a.e. $y \in \mathbb{R}^N$.

7. Concluding remarks. Our proof of the formula (1.9) is based on the following three results:

- (i) *There is $\lambda_0 > 0$ such that $S_\lambda \in B(L_2)$ for all $\lambda \geq \lambda_0$.*
- (ii) *There is $\lambda_0 > 0$ such that $T_\lambda \in B(L_2)$ with $\|T_\lambda\|_{B(L_2)} < 1$ for all $\lambda \geq \lambda_0$.*
- (iii) *The operator H_0 on L_2 is nonnegative and essentially selfadjoint.*

We proved these three results under the assumptions on q stated at the beginning of §1. Let us now assume that q is only positive and continuous on \mathbb{R}^N , and satisfies (1.1). Then the proof of Proposition 1.2 shows that (i) stays valid. If, in addition,

(ii) and (iii) are valid then we still can prove (1.9) using similar arguments as in the proof of Theorem 1.3.

The statement (ii) is valid if the following condition is satisfied: there are constants $\lambda_0 > 0$, $\alpha \in (0, 1]$, $C' > 0$ and $c' \in (0, c)$ such that

$$(7.1) \quad |q(x)^{2m} - q(y)^{2m}| \cdot |x - y|^{-\alpha} \leq C' q(x)^{2m} e^{c' \tau_{\lambda_0}(x)|x-y|},$$

for all $x, y \in \mathbb{R}^N$, $x \neq y$. Here $c > 0$ is the constant from Proposition 8.1. It is easy to see that (7.1) can be substituted for (2.10) in the proof of Proposition 1.2. Then (1.8) has to be replaced by

$$(7.2) \quad \|T_\lambda\|_{B(L_2)} \leq \tilde{C} \lambda^{-\alpha/2m}, \quad \lambda \geq \lambda_0.$$

The statement (iii) is valid if $m = 1$ (cf. Wienholtz [25]), or if q satisfies both (7.1) and

$$(7.3) \quad q(y) \leq C' q(x) e^{C'|x-y|}, \quad x, y \in \mathbb{R}^N.$$

Again, it is easy to see that (7.3) can be substituted for (2.9) in the proof of Theorem 1.3.

Our formula (1.9) can be used to estimate the Hilbert-Schmidt norm of the resolvent $(H + \lambda I)^{-1}$ with arbitrary precision as $\lambda \rightarrow \infty$. Namely, we have the Neumann series

$$(H + \lambda I)^{-1} = \sum_{n=0}^{\infty} T_\lambda^n S_\lambda$$

in $B(L_2)$, and the estimate (1.8). It is easy to compute the Hilbert-Schmidt norm of the operator S_λ to obtain the first approximation, whereas this task may be difficult in the case of $\sum_{n=0}^{n_0} T_\lambda^n S_\lambda$ for n_0 large. In the case $m = 1$, the quantity $\|(H + \lambda I)^{-1}\|_{HS(L_2)}$ was computed by Titchmarsh [23, §17.11] and Otelbaev [15] with precision of order $\lambda^{-1/2}$ as $\lambda \rightarrow \infty$ under slightly weaker conditions than (7.1) and (7.3), but the condition $q^{-\gamma} \in L_1(\mathbb{R}^N)$ for some $\gamma > 0$. We are not aware of any other method than (1.9) which would enable us to compute the quantity $\|(H + \lambda I)^{-1}\|_{HS(L_2)}$ with precision higher than $\lambda^{-1/2m}$ as $\lambda \rightarrow \infty$. As it is shown in the proofs of Theorems 1.4 and 1.5, the formula (1.9) seems to be relevant also for other computations. For instance, we can estimate also the Hilbert-Schmidt norms of the operators $(H + \lambda I)^{-\omega}$ and $A_{\lambda, \omega}^{-1} = M_p^{-1}(H + \lambda I)^{-\omega}$ for $\omega \in (0, 1)$ (cf. Takáč [21, Theorem V.1.1]).

8. Appendix. Let $m \geq 1$ be an integer. We denote by G_{2m} the Fourier transform of the function $[(2\pi|y|)^{2m} + 1]^{-1}$ of $y \in \mathbb{R}^N$.

PROPOSITION 8.1. *We have $G_{2m} \in C^\infty(\mathbb{R}^N \setminus \{0\})$, and there exists two constants $c > 0$ and $C > 0$ such that, for all $x \in \mathbb{R}^N \setminus \{0\}$,*

$$(8.1) \quad |G_{2m}(x)| e^{c|x|} \leq \begin{cases} C|x|^{2m-N} & \text{if } N > 2m; \\ C(1 + |\log|x||) & \text{if } N = 2m; \\ C & \text{if } N < 2m. \end{cases}$$

In particular, the estimate (2.1) is valid.

A proof of this result can be found in Takáč [21, Proposition VII.1.1]. It makes use of standard methods for asymptotics of the Fourier transform of an analytic function (cf. John [8, Chapter III]).

PROPOSITION 8.2. Let ρ be a positive function on \mathbb{R}^N such that $\log \rho$ is uniformly Lipschitz continuous with a Lipschitz constant $C > 0$. If

$$(8.2) \quad \int_{\mathbb{R}^N} \rho(x) dx < \infty,$$

then $\rho(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

PROOF. Making use of (2.9) with ρ in place of q we obtain

$$C' = \sup\{\rho(x+h)/\rho(x) \mid x, h \in \mathbb{R}^N, |h| \leq 1\} < \infty.$$

Let us assume that there exists a sequence $\{x_n\}_{n=1}^\infty$ in \mathbb{R}^N such that $c_0 = \inf\{\rho(x_n) \mid n \geq 1\} > 0$ and $|x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Passing to a subsequence if necessary we may assume that $|x_{n+1}| > |x_n| + 2$, $n \geq 1$. Then we have, for all $n \geq 1$,

$$(8.3) \quad \rho(x_n + h) \geq c_0/C', \quad h \in \mathbb{R}^N, |h| \leq 1.$$

Since the unit balls $B_1(x_n) = \{x \in \mathbb{R}^N \mid |x - x_n| < 1\}$ are pairwise disjoint, (8.3) contradicts (8.2).

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37235